Name: .....

Gosford High School

2022 Trial HSC examination

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–34, show relevant mathematical reasoning and/ or calculations

Total Marks

100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Detach the *Multiple-choice answer sheet* from the last page of this question booklet.

Section II – 90 marks

- Attempt Questions 11–34
- Allow about 2 hours and 45 minutes for this section

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Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet (*provided on the last page of the booklet*) for Questions 1–10.

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1. The probability distribution table for a discrete random variable X is shown

x	0	1	2
$P(X=x)$	0.5	0.4	0.1

What is the expected value $E(X)$?

A 0.5

B 0.6

C 1.0

D 1.5

2. What is the range of $f(x)=1-x^2$?

A $[0,\infty)$

B $[1,\infty)$

C $(-\infty,0]$

D $(-\infty,1]$

3. Which of the following is a primitive of $x+\cos x$?

A $1+\sin x$

B $1-\sin x$

C $\frac{x^2}{2}-\sin x$

D $\frac{x^2}{2}+\sin x$

4. Audrey recently completed her assessment tasks.

The class scores on each test were normally distributed. The table shows the subjects and Audrey's scores as well as the mean and standard deviation of the class scores on each test.

Subject	Audrey's Mark	Mean	Standard deviation
English	75	70	5
Mathematics	65	60	10
Japanese	82	85	2
History	80	76	2

Relative to the rest of the class, which subject did Audrey perform best in?

A English

B Mathematics

C Japanese

D History

5. What are the solutions of $\sqrt{3}\tan x = -1$ for $0 \le x \le 2\pi$?

A $x = \frac{\pi}{3}$ or $x = \frac{2\pi}{3}$

B $x = \frac{\pi}{6}$ or $x = \frac{11\pi}{6}$

C $x = \frac{2\pi}{3}$ or $x = \frac{5\pi}{3}$

D $x = \frac{5\pi}{6}$ or $x = \frac{11\pi}{6}$

6. A bag contains 4 maroon coloured marbles and 6 white coloured marbles. Three marbles are selected at random without replacement.

What is the probability that at least one of the marbles selected is maroon?

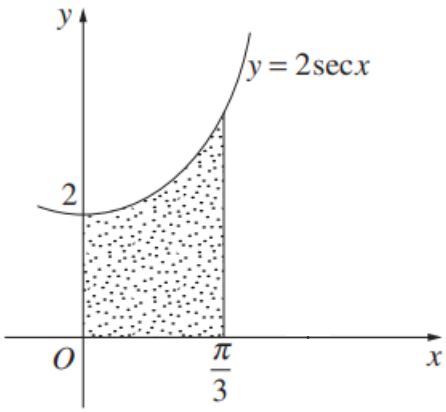
A $\frac{1}{6}$

B $\frac{1}{2}$

C $\frac{5}{6}$

D $\frac{29}{30}$

7. In the diagram, the shaded region is bounded by the curve $y = 2\sec x$, the coordinate axes and the line $x = \frac{\pi}{3}$.



Which inequality best describes the shaded area

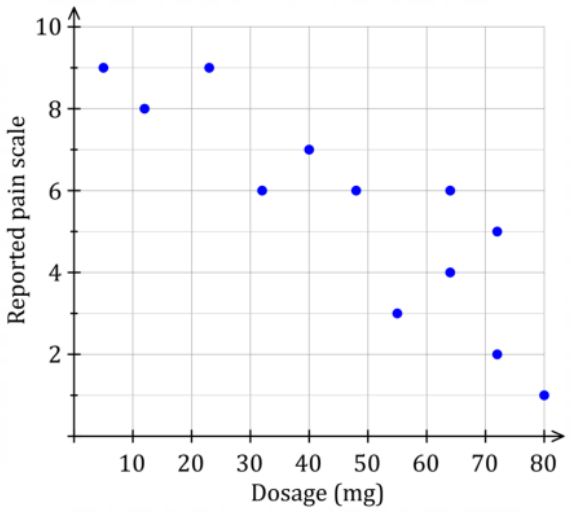
A $\frac{2\pi}{3} < \int_0^{\frac{\pi}{3}} 2\sec x \, dx < \frac{4\pi}{3}$

B $\frac{\pi}{3} < \int_0^{\frac{\pi}{3}} 2\sec x \, dx < 4$

C $0 < \int_0^{\frac{\pi}{3}} 2\sec x \, dx < \frac{2\pi}{3}$

D $\frac{2\pi}{3} < \int_0^{\frac{\pi}{3}} \sec x \, dx < 4$

8. A scatterplot of pain (as reported by patients) compared to the dosage (in mg) of a drug is shown below.



Which of the following is the best estimate for Pearson's correlation coefficient r ?

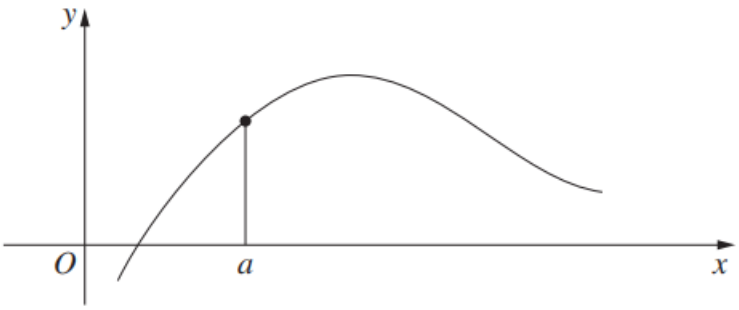
A $r = 0.87$

B $r = -0.87$

C $r = 0.61$

D $r = -0.61$

9. The diagram shows the graph $y = f(x)$



Which of the following statements is true?

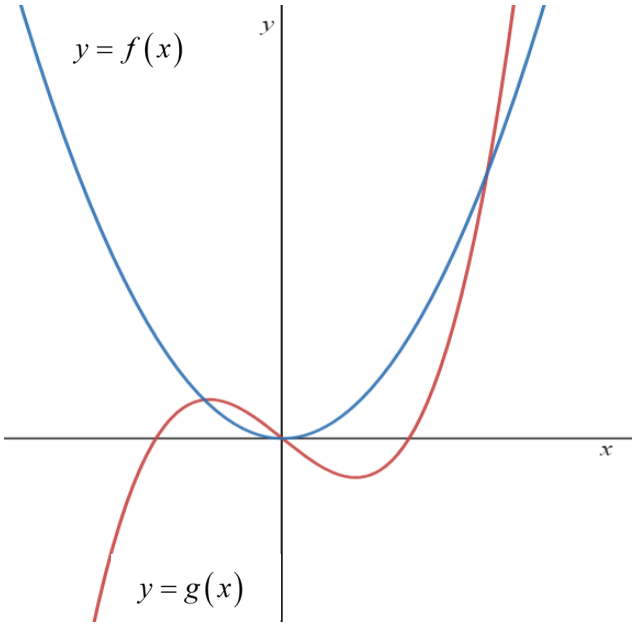
A $f''(a) < f'(a) < 0$

B $0 < f'(a) < f''(a)$

C $0 < f''(a) < f'(a)$

D $f''(a) < 0 < f'(a)$

10 The graph shows two functions $y = f(x)$ and $y = g(x)$



Define $h(x) = f(x) - g(x)$.

How many stationary points does $y = h(x)$ have?

A 0

B 1

C 2

D 3

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Student number

2022 Trial HSC Examination

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

A ⊗ B ⊗ (correct ↓) C ○ D ○

Section 1: Student responses

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

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2022 Gosford High School Trial HSC

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Mathematics Advanced

Section II Answer Booklet 1

Section II

90 marks

Attempt Questions 11–34

Allow about 2 hours and 45 minutes for this section

Booklet 1 — Attempt Questions 11–27 (52 marks)

Booklet 2 — Attempt Questions 28–34 (38 marks)

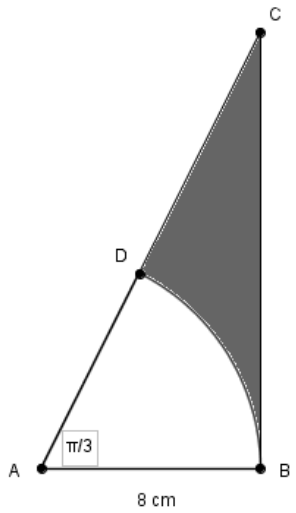
Instructions

- **Write your Student Number at the top of this page.**
- **Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.**
- **Your responses should include relevant mathematical reasoning and/or calculations.**
- **Extra writing space is provided on pages 22-24. If you use this space, clearly indicate which question you are answering**

Question 11 (3 marks)

(a) In the diagram, triangle ABC is right angled at B. AB = 8cm and $\angle BAC = \frac{\pi}{3}$.

The circular arc BD has centre A with radius AB.



(i) Find the exact area of sector ABD. **1**

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(ii) Hence find the exact area of the shaded region BCD. **2**

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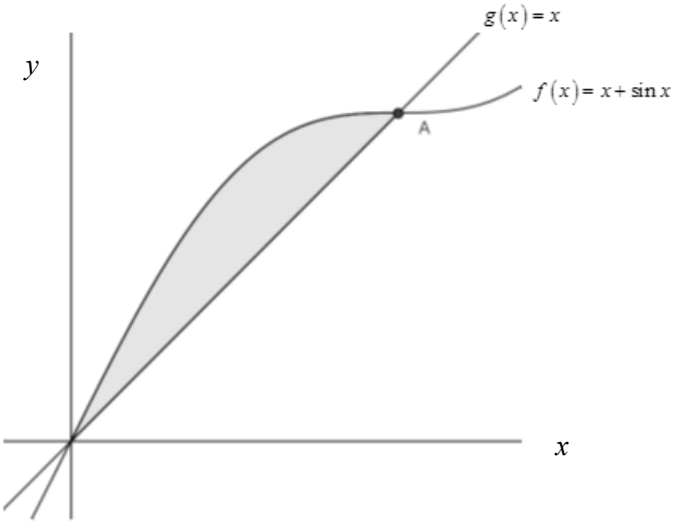
Question 12 (3 marks)

The diagram shows the graphs of $f(x) = x + \sin x$ and $g(x) = x$ intersecting at the origin and point A.



(i) Find the x coordinate of the point A. **1**

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(ii) Hence find the exact area of the shaded region. **2**

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Question 13 (2 marks)

Evaluate $\int_0^e 1 - e^x \, dx$ **2**

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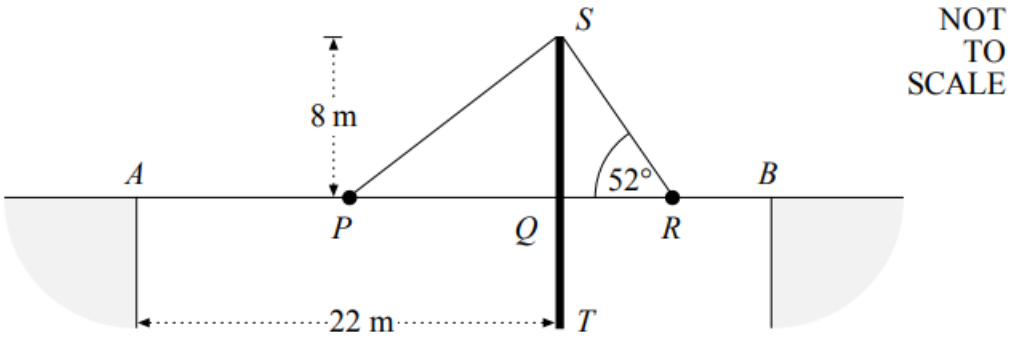
Question 14 (3 marks)

A horizontal bridge is built between points A and B. The bridge is supported by cables SP and SR, which are attached to the top of a vertical pylon ST.



The section of the pylon, SQ, above the bridge is 8 metres long and $\angle SRQ = 52^\circ$.

The distance AQ is 22 metres, and P is the midpoint of AQ

(i) Find the length of the cable SR. **1**

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(ii) Find the size of $\angle SPQ$ to the nearest degree. **2**

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Question 15. (8 marks)

Consider the function $f(x) = 2x^4 - x^5$.

(i) Find $f'(x)$ **1**

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(ii) Find the coordinates of the stationary points of the curve $y = f(x)$, and determine their nature. **3**

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(iii) Find any points of inflection. **2**

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Question 15 continued

(iv) In the space below, sketch the graph of $y = f(x)$ showing the above information and any intercepts. **2**

Question 16. (2 marks)

Given the function $f(x) = x^2 - 2$ and $g(x) = \sqrt{7 - x}$ sketch $y = f(g(x))$ over its natural domain. **2**

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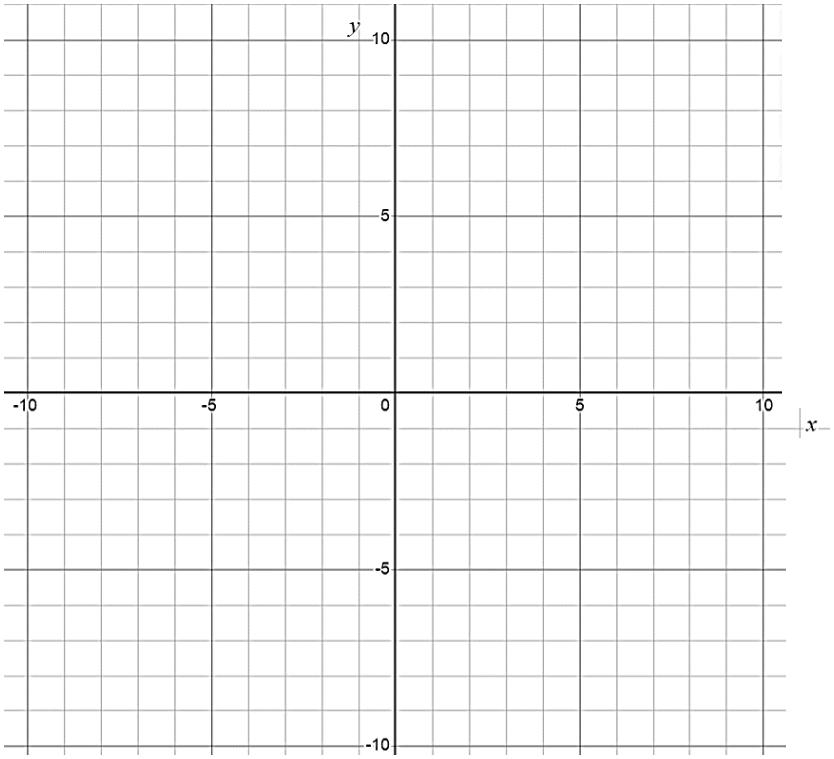

Question 17. (2 marks)

Let $\log_a 2 = x$ and $\log_a 3 = y$.

Find an expression for $\log_a 12$ in terms of x and y. **2**

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Question 18. (6 marks)

Differentiate with respect to x

(i) $x \tan x$ **2**

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(ii) $\sin x \cos^2 x$ **2**

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(iii) $\dfrac{x}{e^x}$ **2**

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Question 19. (4 marks)

Coal is extracted from a mine at a rate that is proportional to the amount of coal remaining in the mine. Hence the amount R remaining after t years is given by $R = R_0 e^{-kt}$, where k is a constant and R_0 is the initial amount of coal.

After 20 years, 50% of the initial amount of coal remains.

(i) Find the exact value of k. **2**

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(ii) How many more years will elapse before only 10% of the original amount remains? **2**

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Question 20. (2 marks)

A student was asked to differentiate $f(x) = x^2 - 2x$ from first principles. The student began the solution as shown below. Complete the solution. **2**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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Question 21. (2 marks)

Loudness, in decibels, is given by the formula $L = \log_e\left(\frac{I}{I_0}\right)$, where I is the intensity level and I_0 is the threshold sound (or sound that can barely be heard). Sounds louder than 85 dB can damage hearing.

The intensity of a vacuum cleaner is estimated to be 10 000 000 times the threshold level I_0. Would using a vacuum cleaner damage your hearing based on this estimate? Justify your answer with appropriate working. **2**

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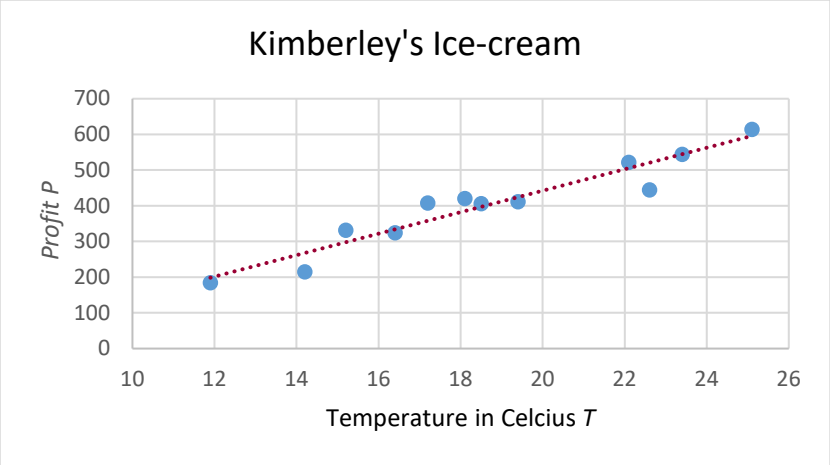
Question 22. (2 marks)

The graph shows Kimberley's profit from ice-cream sales over the last 12 days.



The equation of the line of best fit is $P = 28.6T - 130$, where P is the profit and T the temperature in Celsius.

The correlation coefficient is $r = 0.95$

(i) Describe the correlation. **1**

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(ii) By using the equation of the line of best fit, predict the profit for a day whose temperature is $20^\circ C$ **1**

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Question 23. (2 marks)

Five values of the function $f(x)$ are shown in the table. **2**

x	0	5	10	15	20
$f(x)$	15	25	22	18	10

Use the trapezoidal rule with the five function values given in this table to estimate $\int_0^{20} f(x)\,dx$.

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Question 24. (3 marks)

Greg lives in Gosford and is starting a new job in Sydney. He needs to catch a train to get to work and will be late if the trains do not run to schedule. The probability that, on any given day, his train will run to schedule is 0.96.

(i) What is the probability that Greg's train is late on the first day? **1**

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(ii) What is the probability that Greg arrives late on exactly one of the first three days of his new job? **2**

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Question 25. (3 marks)

Solve $\log_e x - \dfrac{3}{\log_e x} = 2$ giving your answers in exact form. **3**

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Question 26. (2 marks)

Show that $\dfrac{\sec^2 x}{\tan^2 x} = \operatorname{cosec}^2 x$ **2**

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Question 27. (3 marks)

The function $f(x) = 2 - 2^x$ is dilated vertically by a factor of 2 and shifted 4 units to the right.

Find the equation of the function formed and sketch the result showing key features. **3**

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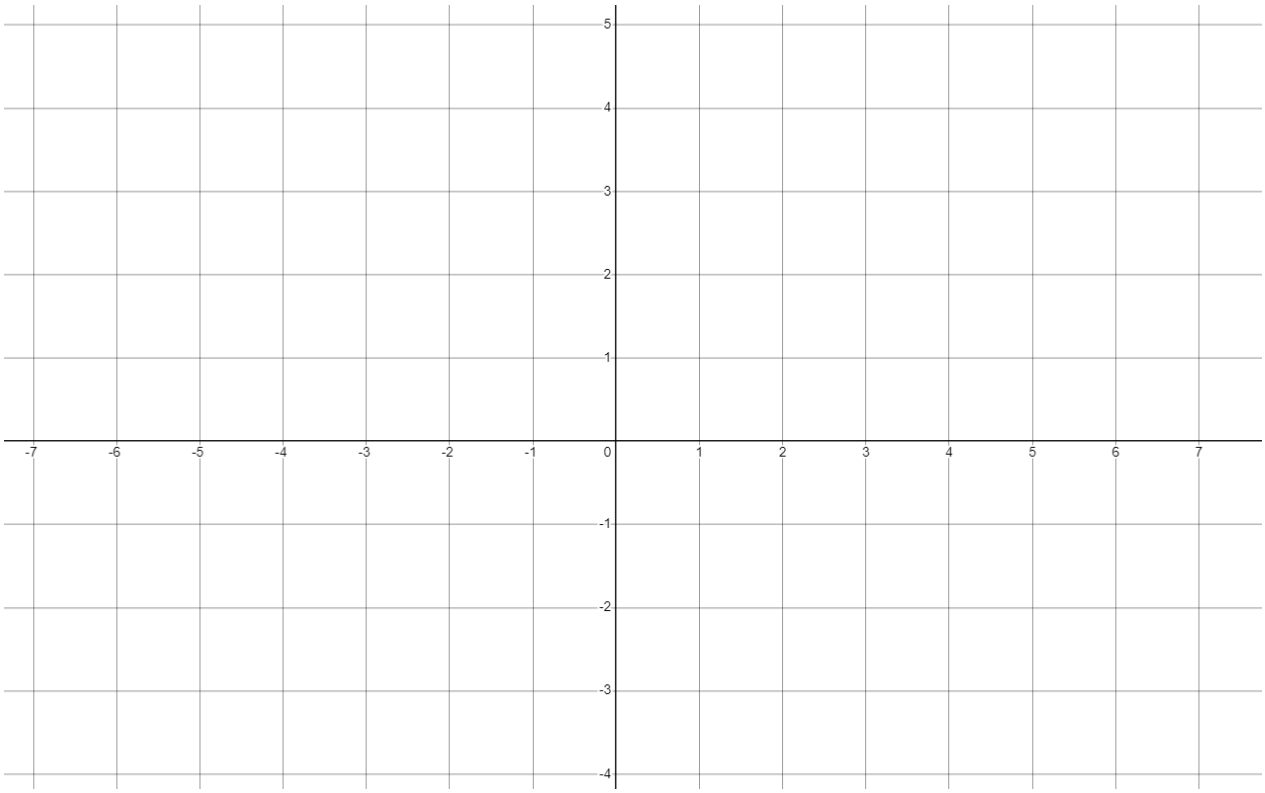

Section II Booklet 1 - extra writing space

If you use this space, clearly indicate which question you are answering.

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2022 **Gosford High School Trial HSC**

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Mathematics Advanced

Section II Answer Booklet 2

Booklet 2 — Attempt Questions 28–34 (38 marks)

Instructions

- **Write your Student Number at the top of this page.**
- **Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.**
- **Your responses should include relevant mathematical reasoning and/or calculations.**
- **Extra writing space is provided on page 33-34. If you use this space, clearly indicate which question you are answering**

Question 28. (3 marks)

A particle moves in a straight line. At time t seconds, its distance x metres from a fixed point O on the line is given by $x = 2\sin(3t)$

(i) Sketch $x = 2\sin(3t)$ for $0 \le t \le 2\pi$ **2**

(ii) Find the first time when the particle is at rest. **1**

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Question 29. (8 marks)

The queueing time, X minutes, of a teacher waiting on the phone with the Department of Education has a probability density function

$$f(x) = \begin{cases} \frac{3}{32}x(k-x) & 0 \le x \le k \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that the value of k is 4 **3**

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Question 29 continued.

(ii) Sketch the probability density function $f(x)$ **2**

(iii) What is the mode of the probability density function? **1**

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(iv) Find the probability that the phone will be answered within the first minute. **2**

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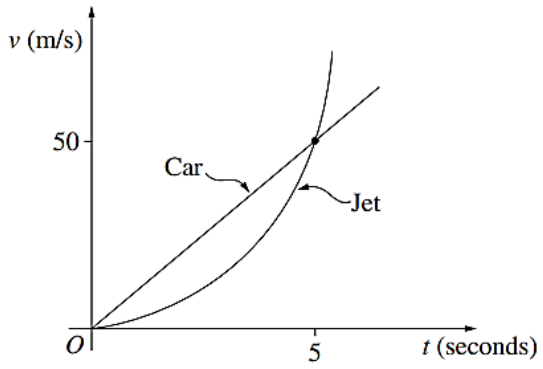
Question 30 (8 marks)



NOT TO SCALE

A car and a jet plane race one another from rest down a runway. The car increases its speed v_1 at a constant rate, while the speed of the jet is given by $v_2 = 2t^2$. After 5 seconds the car and the jet have the same speed of 50 m/s, as shown on the graph.

(i) By considering the difference quotient, find the average acceleration of the jet during the 5th second. **2**

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(ii) Find an equation for the speed v_1 of the car in terms of t. **1**

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(iii) For what times is the acceleration of the car greater than that of the jet? **2**

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(iv) Who is winning the race after 5 seconds and by how many metres? **3**

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Question 31 (4 marks)

The lifetime of a particular make of LED bulb is normally distributed with mean of 5000 hours and standard deviation of 425 hours.

(i) Find the z score for a LED with a lifetime of 5374 hours. **1**

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(ii) By considering z scores and consulting the Normal Distribution table below, find the probability that a LED chosen at random has a lifetime between 4898 and 5374 hours. **3**

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891

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Question 32. (5 marks)

The continuous random variable X has the following probability density function

$$f(x) = \begin{cases} a + bx & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $10a + 25b = 2$ **2**

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(ii) If $E(X) = \frac{35}{12}$, find the values for a and b. **3**

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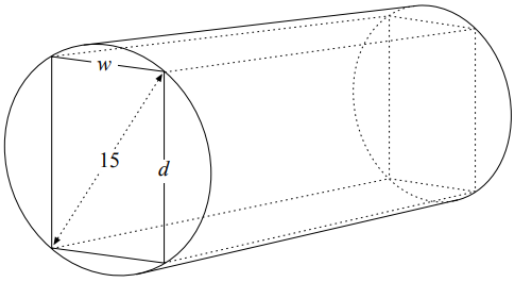
Question 33. (4 marks)



A rectangular beam of width w cm and depth d cm is cut from a cylindrical pine log as shown. The diameter of the cross-section of the log (and hence the diagonal of the cross-section of the beam) is 15 cm. The strength S of the beam is proportional to the product of its width and the square of its depth, so that $S = kwd^2$.

Find the maximum strength of the beam, showing why it is a maximum. **4**

Question 34 (6 marks)

The continuous random variable X has the probability density function $f(x)$ given by

$$f(x) = \begin{cases} \frac{1}{9}\left(x^2 - 2x + 2\right) & 0 \le x \le 3 \\ \frac{1}{3} & 3 < x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(i) In the space provided, sketch the cumulative distribution function $F(x)$ **4**

(ii) Hence, or otherwise, find the 75th percentile of X. **2**

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End of paper

Section II Booklet 2 - extra writing space

If you use this space, clearly indicate which question you are answering.

Gosford High School

2022 Trial HSC examination

Mathematics Advanced

Solutions

2022 Trial HSC Examination

Section I – Multiple Choice Answer Sheet

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Section 1: Solutions

1.	A	○	B	●	C	○	D ○
2.	A	○	B	○	C	○	D ●
3.	A	○	B	○	C	○	D ●
4.	A	○	B	○	C	○	D ●
5.	A	○	B	○	C	○	D ●
6.	A	○	B	○	C	●	D ○
7.	A	●	B	○	C	○	D ○
8.	A	○	B	●	C	○	D ○
9.	A	○	B	○	C	○	D ●
10.	A	○	B	○	C	●	D ○

Section I (10 marks)

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1. The probability distribution table for a discrete random variable X is shown

x	0	1	2
$P(X=x)$	0.5	0.4	0.1

What is the expected value $E(X)$?

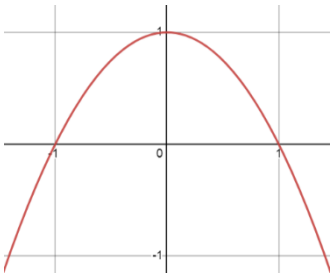
Solution

B 0.6

$$E(X) = 0\times 0.5 + 1\times 0.4 + 2\times 0.1$$
$$= 0.6$$

2. What is the range of $f(x) = 1 - x^2$?

D $(-\infty, 1]$



3. Which of the following is a primitive of $x + \cos x$?

D $\frac{x^2}{2} + \sin x$

4. Audrey recently completed her assessment tasks.

The class scores on each test were normally distributed. The table shows the subjects and Audrey's scores as well as the mean and standard deviation of the class scores on each test.

Subject	Audrey's Mark	Mean	Standard deviation
English	75	70	5
Mathematics	65	60	10
Japanese	82	85	2
History	80	76	2

Relative to the rest of the class, which subject to Audrey perform best in?

D History

In terms of standard deviations

English is $+\sigma$
Mathematics $+0.5\sigma$
Japanese -1.5σ
History $+2\sigma$

Therefore her best subject is
History

5. What are the solutions of $\sqrt{3}\tan x = -1$ for $0 \le x \le 2\pi$?

D $x = \frac{5\pi}{6}$ or $x = \frac{11\pi}{6}$

$$\sqrt{3}\tan x = -1$$
$$\tan x = -\frac{1}{\sqrt{3}}$$
$$x = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6} \text{ for } 0 \le x \le 2\pi$$

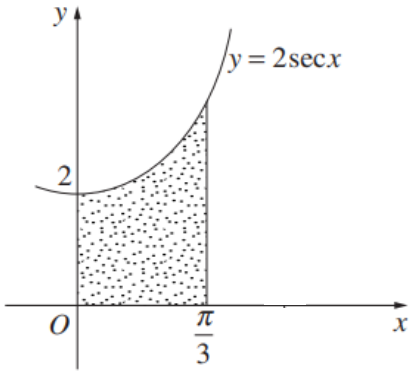
6. A bag contains 4 maroon coloured marbles and 6 white coloured marbles. Three marbles are selected at random without replacement.

What is the probability that at least one of the marbles selected is maroon?

C $\frac{5}{6}$

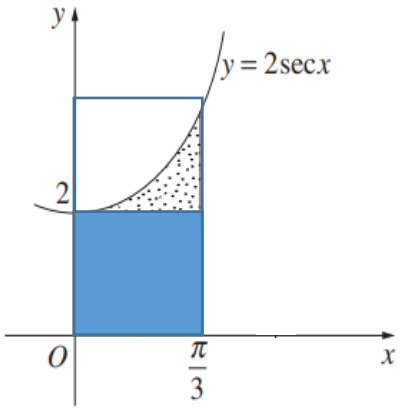
$$P(\text{at least one marble is maroon}) = 1 - P(\text{no marbles are maroon})$$
$$= 1 - \left(\frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}\right)$$
$$= \frac{5}{6}$$

7. In the diagram, the shaded region is bounded by the curve $y = 2\sec x$, the coordinate axes and the line $x = \frac{\pi}{3}$.



Which inequality best describes the shaded area

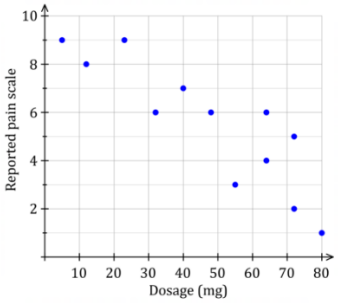
A $\frac{2\pi}{3} < \int_0^{\frac{\pi}{3}} 2\sec x \, dx < \frac{4\pi}{3}$



$A_{\text{lower rectangle}} < A_{\text{shaded}} < A_{\text{upper rectangle}}$

$\frac{\pi}{3} \times 2 < \int_0^{\frac{\pi}{3}} 2\sec x \, dx < \frac{\pi}{3} \times 4$

8. A scatterplot of pain (as reported by patients) compared to the dosage (in mg) of a drug is shown below.

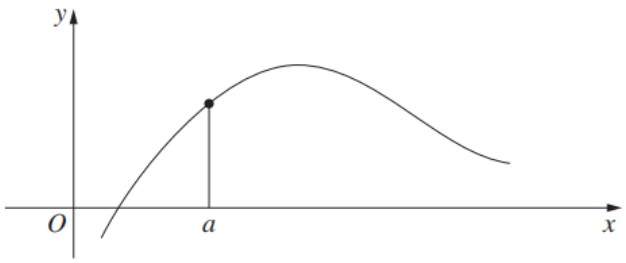


Which of the following is the best estimate for Pearson's correlation coefficient r ?

B $r = -0.87$

Strong negative correlation
Also found by entering data into the calculator

9. The diagram shows the graph $y = f(x)$

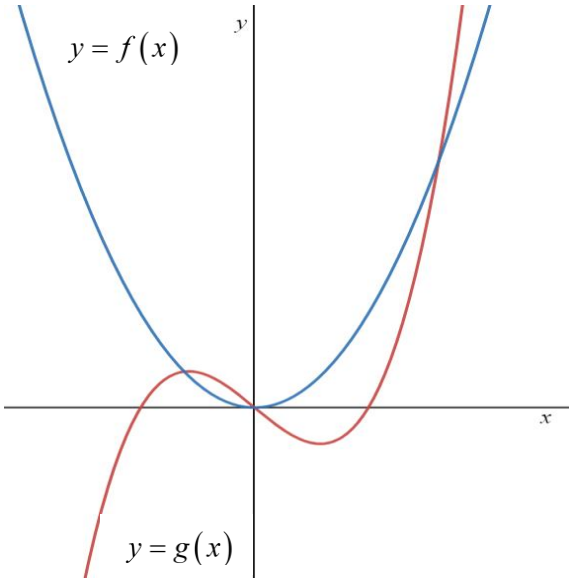


Which of the following statements is true?

D $f''(a) < 0 < f'(a)$ Monotonic increasing at $x = a \Rightarrow f'(a) > 0$

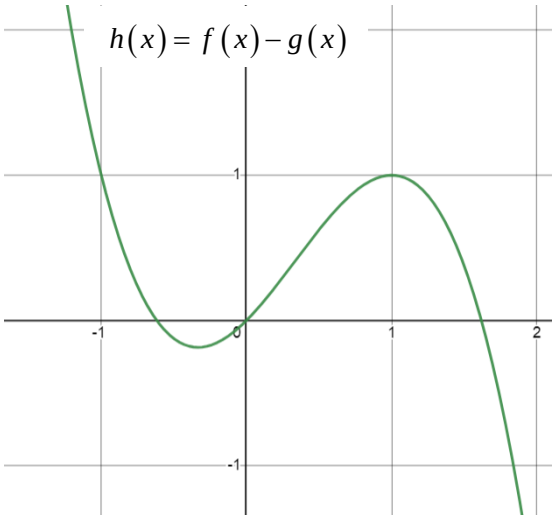
Concave down $x = a \Rightarrow f''(a) < 0$

10 The graph shows two functions $y = f(x)$ and $y = g(x)$



Define $h(x) = f(x) - g(x)$.

How many stationary points does $y = h(x)$ have?



C 2

2022 Gosford High School Trial HSC

Mathematics Advanced

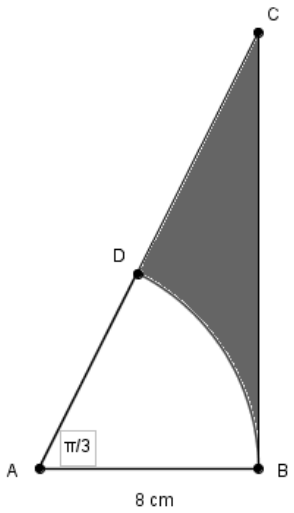
Section II Booklet 1

Solutions

Question 11 (3 marks)

(a) In the diagram, triangle *ABC* is right angled at *B*. AB = 8cm and $\angle BAC = \frac{\pi}{3}$.

The circular arc *BD* has centre *A* with radius *AB*.



(i) Find the exact area of sector *ABD*.

Solution

$$A = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}(8)^2 \times \frac{\pi}{3}$$
$$= \frac{32\pi}{3}$$

Marks	Guideline
1	Correct answer (accept simplified response)
Marker's Comment	

(ii) Hence find the exact area of the shaded region *BCD*.

Solution

First find *BC*

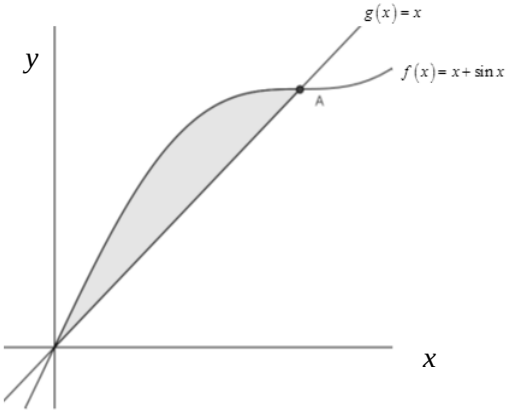
$$\tan\frac{\pi}{3} = \frac{BC}{8}$$
$$BC = 8\tan\frac{\pi}{3}$$

Marks	Guideline
2	Correct response
1	For finding *BC* or CTE
Marker's Comment	

$$A_{\text{Shaded region}} = A_{\text{triangle}} - A_{\text{sector}}$$
$$= \frac{1}{2} \times 8 \times 8\tan\frac{\pi}{3} - \frac{32\pi}{3}$$
$$= 32\sqrt{3} - \frac{32\pi}{3}$$

Question 12 (3 marks)

The diagram shows the graphs of $f(x) = x + \sin x$ and $g(x) = x$ intersecting at the origin and point A.



(i) Find the x coordinate of the point A.

Solution

$x + \sin x = x$

$\sin x = 0$

$x = 0 + k\pi$ where $k \in \mathbb{Z}$

Marks	Guideline
1	Correct answer
Marker's Comment	

From the diagram x is the first positive solution > 0

$x = \pi$

(ii) Hence find the exact area of the shaded region.

Solution

$$A = \int_0^{\pi} f(x) - g(x)\, dx$$

$$= \int_0^{\pi} (x + \sin x) - x\, dx$$

$$= \int_0^{\pi} \sin x\, dx$$

$$= \left[-\cos x\right]_0^{\pi}$$

$$= \left[(-\cos\pi) - (-\cos 0)\right]$$

$$= (-1 \times (-1)) - (-1)$$

$$= 2 \quad \text{units}^2$$

Marks	Guideline
2	Correct response
1	One error in the solution
Marker's Comment	

Question 13 (2 marks)

Evaluate $\int_0^e 1-e^x \, dx$

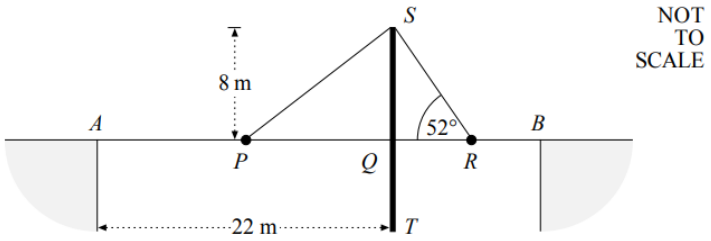
Solution

$$\begin{aligned}\int_0^e 1-e^x \, dx &= \left[x-e^x\right]_0^e \\ &= \left[\left(e-e^e\right)-\left(0-e^0\right)\right] \\ &= e-e^e+1\end{aligned}$$

Marks	Guideline
2	Correct response
1	One error in the solution
Marker's Comment	

Question 14 (3 marks)

A horizontal bridge is built between points *A* and *B*. The bridge is supported by cables *SP* and *SR*, which are attached to the top of a vertical pylon *ST*.



The section of the pylon, *SQ*, above the bridge is 8 metres long and $\angle SRQ = 52^\circ$.

The distance *AQ* is 22 metres, and *P* is the midpoint of *AQ*

(i) Find the length of the cable *SR*.

Solution

$$\begin{aligned}\sin 52^\circ &= \frac{SQ}{SR} \\ SR &= \frac{8}{\sin 52^\circ} \\ &\approx 10.15 \text{ m}\end{aligned}$$

Marks	Guideline
1	Correct answer
Marker's Comment	

(ii) Find the size of $\angle SPQ$ to the nearest degree.

Solution

From the question *P* is the midpoint of *AQ* $i \Rightarrow PQ = 11$ m

$$\begin{aligned}\tan\left(\angle SPQ\right) &= \frac{8}{11} \\ \angle SPQ &= \tan^{-1}\frac{8}{11} \\ &\approx 36^\circ \text{ to the nearest degree}\end{aligned}$$

Marks	Guideline
2	Correct response
1	Recognises tangent ratio but applies incorrect values
Marker's Comment	

Question 15. (8 marks)

Consider the function $f(x) = 2x^4 - x^5$.

(i) Find $f'(x)$

Solution

$$f'(x) = 8x^3 - 5x^4$$

Marks	Guideline
1	Correct answer
Marker's Comment	Question extremely well done.

(ii) Find the coordinates of the stationary points of the curve $y = f(x)$, and determine their nature.

Solution

Stationary points occur when $f'(x) = 0$

$$8x^3 - 5x^4 = 0$$
$$x^3(8 - 5x) = 0$$
$$x = 0 \text{ or } x = \frac{8}{5}$$

Determining the nature

x	-1	0	1	$\frac{8}{5}$	2
$\frac{dy}{dx}$	-12	0	3	0	-16

Marks	Guideline
3	Correct response
2	One arithmetic error or omits *y* ordinate
1	Finds the *x*-ordinates but does not justify the nature
Marker's Comment	Finding x-values of stationary points was done well by most. Students using the second derivative to determine nature often didn't come to the correct conclusion when f''(x)=0.

Therefore there is a minimum turning point at $(0,0)$

And a maximum turning point at $\left(\frac{8}{5}, 2.62\right)$

(iii) Find any points of inflection.

Possible points of inflection occur when $f''(x) = 0$

$$f''(x) = 24x^2 - 20x^3$$
$$24x^2 - 20x^3 = 0$$
$$4x^2(6 - 5x) = 0$$
$$x = 0 \text{ or } x = \frac{6}{5}$$

Testing for inflection. Noting $x = 0$ is a minimum point

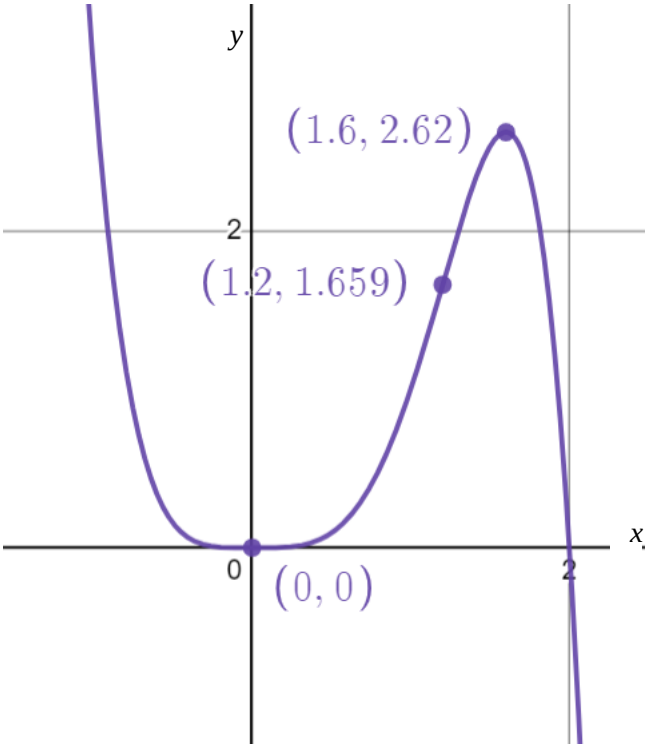
x	1	$\frac{6}{5}$	2
$\frac{d^2y}{dx^2}$	4	0	-64

Marks	Guideline
2	Correct response
1	Finds the point of inflection, without confirming the inflection or with error
Marker's Comment	Good students recognised that x=0 could not be a POI if they'd correctly found it's nature in the previous section. All possible POI should be tested for changes in concavity. Many students did not recognise or attempt to find POI's using the 2nd derivative.

Since there is a change in concavity, there is a point of inflection at $\left(\frac{6}{5}, 1.659\right)$

Question 15 continued

(iv) In the space below, sketch the graph of $y = f(x)$ showing the above information and any intercepts.



Marks	Guideline
2	Correct response
1	Omits labels or problems with the sketch
Marker's Comment	Students who found incorrect points often found it difficult to connect them in a logical way. Most students successfully found x-intercepts.

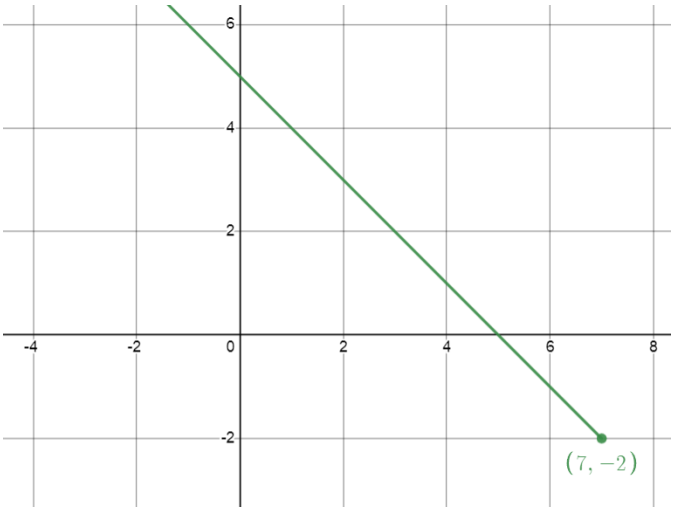
Question 16. (2 marks)

Given the function $f(x) = x^2 - 2$ and $g(x) = \sqrt{7 - x}$ sketch $y = f(g(x))$ over its natural domain.

Solution

The domain of $g(x)$ is $x \le 7 \Rightarrow$ the domain of $y = f(g(x))$ is $x \le 7$

$$f(x) = \left(\sqrt{7 - x}\right)^2 - 2$$
$$= 7 - x - 2$$
$$= 5 - x$$



Marks	Guideline
2	Correct response
1	Incorrect domain
Marker's Comment	Many students did not recognise the restricted domain. Important for students to clearly represent on their graph that the point (7,-2) is included using a larger, solid dot.

Question 17. (2 marks)

Let $\log_a 2 = x$ and $\log_a 3 = y$.

Find an expression for $\log_a 12$ in terms of x and y.

Solution

$$\begin{aligned}\log_a 12 &= \log_a 4 \times 3 \\ &= \log_a 4 + \log_a 3 \\ &= \log_a 2^2 + \log_a 3 \\ &= 2\log_a 2 + \log_a 3 \\ &= 2x + y\end{aligned}$$

Marks	Guideline
2	Correct response
1	Can show 1 correct log law related to the question
Marker's Comment	Many students confused with the Log Laws, often multiplying when they should add.

Question 18. (6 marks)

Differentiate with respect to x

(i) $x\tan x$

Solution

$$\begin{aligned}\frac{d\left(x\tan x\right)}{dx} &= x\times\left[\sec^2 x\right] + \tan x\left[1\right] \\ &= x\sec^2 x + \tan x\end{aligned}$$

Marks	Guideline
2	Correct response
1	Can show 1 correct derivative related to the question
Marker's Comment	Most students answered well using the product rule. Students who struggled tried to differentiate without using a rule.

(ii) $\sin x\cos^2 x$

Solution

$$\begin{aligned}\frac{d\left(\sin x\cos^2 x\right)}{dx} &= \sin x\left[2\cos x\left(-\sin x\right)\right] + \cos^2 x\left[\cos x\right] \\ &= -2\sin^2 x\cos x + \cos^3 x \\ &= \cos x\left(\cos^2 x - 2\sin^2 x\right)\end{aligned}$$

Marks	Guideline
2	Correct response
1	Can demonstrate the product rule but with error in the chain rule
Marker's Comment	Many students confused with the derivative of $\cos^2 x$. Some students used substitution to avoid using the chain rule. Many different forms of the answer accepted.

(iii) $\dfrac{x}{e^x}$

$$\begin{aligned}\frac{d\left(\frac{x}{e^x}\right)}{dx} &= \frac{e^x\left[1\right] - x\left[e^x\right]}{\left(e^x\right)^2} \\ &= \frac{e^x\left(1-x\right)}{e^{2x}} \\ &= \frac{1-x}{e^x}\end{aligned}$$

Marks	Guideline
2	Correct response
1	Can demonstrate the product rule but with error in the chain rule
Marker's Comment	Done well in general. Some students mixing up u and v in use of quotient rule. Students should represent answers in simplest form using appropriate factorisation and cancelling.

Question 19. (4 marks)

Coal is extracted from a mine at a rate that is proportional to the amount of coal remaining in the mine. Hence the amount *R* remaining after *t* years is given by $R = R_0 e^{-kt}$, where *k* is a constant and R_0 is the initial amount of coal. After 20 years, 50% of the initial amount of coal remains.

(i) Find the exact value of *k*.

Solution

When $t = 20, R = \frac{1}{2}R_0$

$$\frac{1}{2}R_0 = R_0 e^{-k(20)}$$

$$\ln\left(\frac{1}{2}\right) = \ln e^{-20k}$$

$$-20k = -\ln 2$$

$$k = \frac{1}{20}\ln 2$$

Marks	Guideline
2	Correct response (accept different forms of *k*)
1	Substituted initial values into the equation
Marker's Comment	Better responses substituted R with ½ R_0. Other responses substituted specific values for R and R_0 and were still able to work towards a solution. Many responses went on to approximate the value of k which indicates that they either didn't recognise that the question asked for the exact value, or they didn't understand what "exact value" meant. Students who approximated k did not receive full marks.

(ii) How many more years will elapse before only 10% of the original amount remains?

Solution

$$0.1R_0 = R_0 e^{-\frac{1}{20}\ln 2 \times t}$$

$$0.1 = e^{-\frac{1}{20}\ln 2 \times t}$$

$$\ln(0.1) = \ln e^{-\frac{1}{20}\ln 2 \times t}$$

$$-\frac{1}{20}\ln 2 \times t = -\ln 10$$

$$t = \frac{20\ln 10}{\ln 2}$$

$$\approx 66.43$$

Marks	Guideline
2	Correct response
1	Found the correct value for *t*
Marker's Comment	Few students recognised the significance of the word "more" in the question. Responses that did not incorporate the first 20 years into their answer were not awarded fully marks.

However, the question asks for how many *more* years $= 66.43 - 20$

$= 40.43$ more years

Question 20. (2 marks)

A student was asked to differentiate $f(x) = x^2 - 2x$ from first principles. The student began the solution as shown below. Complete the solution.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^2 - 2(x+h) - \left[x^2 - 2x\right]}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + 2hx + h^2 - 2x - 2h - x^2 + 2x}{h}$$

$$= \lim_{h\to 0}\frac{2hx + h^2 - 2h}{h}$$

$$= \lim_{h\to 0} 2x - 2 + h$$

$$= 2x - 2$$

Marks	Guideline
2	Correct response
1	Found the first line and or skipped steps
Marker's Comment	This question was reasonably well attempted. The most common errors were neglecting to put brackets around f(x), or only partially substituting (x+h) for x. Students who found the derivative without deriving from first principles did not receive any marks. Students should be aware that the limit notation is fundamental to the process of differentiating from first principles as it describes how to deal with the *h* term. Neglecting this notation in solutions prevented some students from receiving full marks.

Question 21. (2 marks)

Loudness, in decibels, is given by the formula $L = \log_e\left(\frac{I}{I_0}\right)$, where I is the intensity level and I_0 is the threshold sound (or sound that can barely be heard). Sounds louder than 85 dB can damage hearing.

The intensity of a vacuum cleaner is estimated to be 10 000 000 times the threshold level I_0. Would using a vacuum cleaner damage your hearing based on this estimate? Justify your answer with appropriate working.

Solution

Decibels L on the vacuum cleaner

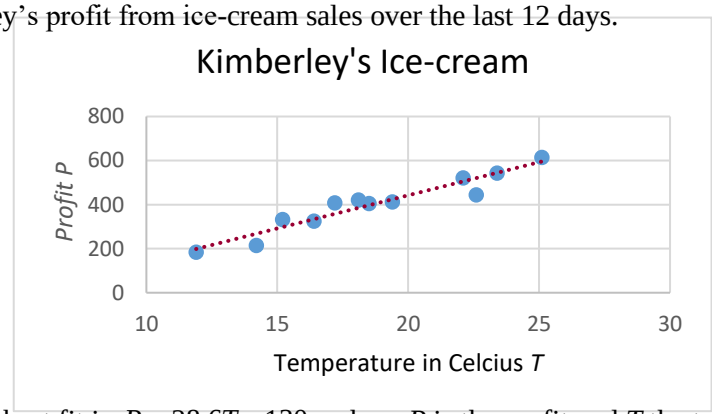
$$L = \log_e\left(\frac{10\,000\,000 \times I_0}{I_0}\right)$$
$$= \ln(10\,000\,000)$$
$$\approx 16.11 \text{ db}$$

Marks	Guideline
2	Correct response
1	Found the decibels but did not answer the question
Marker's Comment	This question was mostly well understood.

Therefore, this vacuum cleaner would not damage your hearing.

Question 22. (2 marks)

The graph shows Kimberley's profit from ice-cream sales over the last 12 days.



The equation of the line of best fit is $P = 28.6T - 130$, where P is the profit and T the temperature in Celsius.

The correlation coefficient is $r = 0.95$

(i) Describe the correlation.

Solution

Since the Pearson's correlation coefficient is $r = 0.95$,

There is a strong positive correlation between profit and temperature.

Marks	Guideline
1	Correct answer
Marker's Comment	Both strength and direction of the correlation were required to achieve the mark for this question.

(ii) By using the equation of the line of best fit, predict the profit for a day whose temperature is $20^\circ C$

Solution

$$P = 28.6(20) - 130$$
$$= \$442$$

Marks	Guideline
1	Correct answer
Marker's Comment	Students must have used the equation for the line of best fit to obtain the answer. Many students made an approximation from the graph, but the question clearly indicated that the equation should have been used.

Question 23. (2 marks)

Five values of the function $f(x)$ are shown in the table.

x	0	5	10	15	20
$f(x)$	15	25	22	18	10

Use the trapezoidal rule with the five function values given in this table to estimate $\int_0^{20} f(x)\,dx$.

Solution

$$\int_0^{20} f(x)\,dx = \frac{\frac{20-0}{4}}{2}\left[f(0)+f(20)+2\left(f(5)+f(10)+f(15)\right)\right]$$
$$= \frac{5}{2}\left[15+10+2(25+22+18)\right]$$
$$= 387.5$$

Marks	Guideline
2	Correct response
1	Applied the formula with error
Marker's Comment	Was not done as well as expected. When asked to use a specific rule then students should show evidence of using that rule. Students need to remember that 'n' stands for the number of sub-intervals and not the number of function values. Careless errors were made mixing up x-vales and function values.

Question 24. (3 marks)

Greg lives in Gosford and is starting a new job in Sydney. He needs to catch a train to get to work and will be late if the trains do not run to schedule. The probability that, on any given day, his train will run to schedule is 0.96.

(i) What is the probability that Greg's train is late on the first day?

Solution

$$P(\text{Greg is late}) = 1-0.96$$
$$= 0.04$$

Marks	Guideline
1	Correct answer
Marker's Comment	Well done. Most students showed evidence of using complementary probability by subtracting the on time probability from 1.

(ii) What is the probability that Greg arrives late on exactly one of the first three days of his new job?

Solution

$$P(\text{late on exactly one day}) = 0.04\times 0.96^2 + 0.96\times 0.04\times 0.96 + 0.96^2\times 0.04$$
$$\approx 0.11$$

Marks	Guideline
2	Correct response
1	Some correct working
Marker's Comment	Better responses included a tree diagram to show that there were 3 different ways of arriving late on exactly 1 day. Many students only showed probability product of 2 lates and one on time, without considering that the on time day could occur on Day1 or Day 2 or Day 3, requiring the students to multiply their answer by 3.

Question 25. (3 marks)

Solve $\log_e x - \frac{3}{\log_e x} = 2$ giving your answers in exact form.

Marks	Guideline
3	Correct response
2	One arithmetic error
1	Recognise that the equation can be considered a quadratic
Marker's Comment	Most students got full marks or no marks. Care needs to be taken when multiplying

Solution

$$\log_e x - \frac{3}{\log_e x} = 2$$

$$\left(\log_e x\right)^2 - 3 = 2\log_e x$$

$$\left(\log_e x\right)^2 - 2\log_e x - 3 = 0$$

$$\left(\log_e x - 3\right)\left(\log_e x + 1\right) = 0$$

$$\log_e x = 3 \quad \text{or} \quad \log_e x = -1$$

$$x = e^3 \qquad x = \frac{1}{e}$$

	$\log_e x$ by itself, with students getting $\log_e x^2$ instead of $\left(\log_e x\right)^2$. Better responses let $\log_e x = u$ which easily created a quadratic to solve. A hard application of this content area.

Question 26. (2 marks)

Show that $\dfrac{\sec^2 x}{\tan^2 x} = \operatorname{cosec}^2 x$

Solution

$$\begin{aligned} LHS &= \frac{\sec^2 x}{\tan^2 x} \\ &= \frac{\dfrac{1}{\cos^2 x}}{\dfrac{\sin^2 x}{\cos^2 x}} \\ &= \frac{1}{\sin^2 x} \\ &= \operatorname{cosec}^2 x \end{aligned}$$

Marks	Guideline
2	Correct response
1	Some correct working
Marker's Comment	Well done. Students successfully substituted two trig results and then simplified to get the RHS.

Question 27. (3 marks)

The function $f(x) = 2 - 2^x$ is dilated vertically by a factor of 2 and shifted 4 units to the right.

Find the equation of the function formed and sketch the result showing key features.

Solution

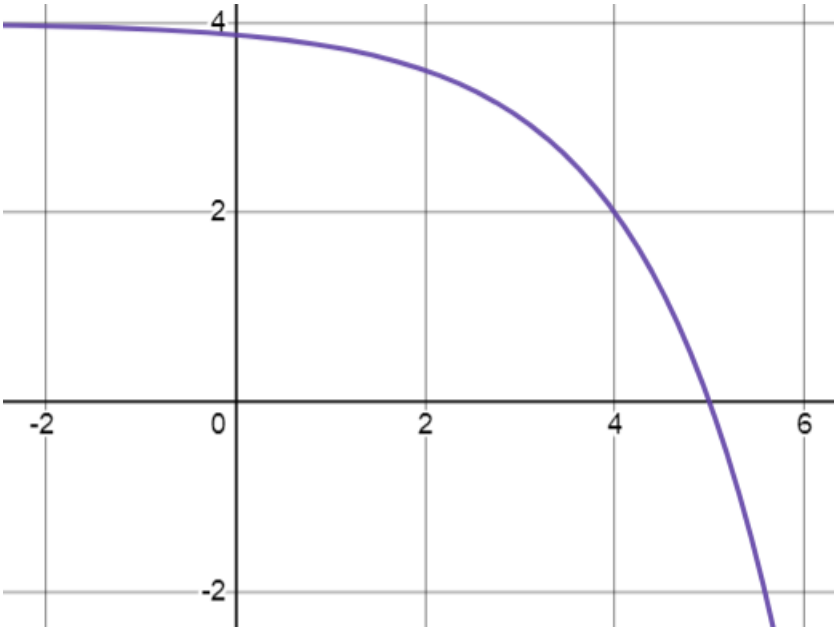
Dilating vertically by a factor of 2

$$\frac{f(x)}{2} = 2 - 2^x$$
$$f(x) = 4 - 2.2^x$$

Shifting right by 4 units

$$\begin{aligned} f(x-4) &= 4 - 2.2^{x-4} \\ &= 4 - 2^{x-3} \end{aligned}$$

Marks	Guideline
3	Correct response (accept different forms of the function)
2	One error (i.e. CTE)
1	One correct element
Marker's Comment	Most students realised to do the horizontal translation you subtracted 4 from x, but the vertical dilation needed to be applied to the whole function and not just x or 2^x. There were many Index Law errors, students need to remember that when you are multiplying and the bases are the same, you add the powers but the bases stay the same. Maybe more practice on Index Laws with numeric bases. When sketching graphs, key features include x and y intercepts. Also, exponential graphs should have a asymptote somewhere. Students' graphs should match there answer for the Transformations. Though question was not a difficult application, the content can be confusing. Students need to learn a process and know how to apply it.



2022 **Gosford High School Trial HSC**

Mathematics Advanced

Section II Answer Booklet 2

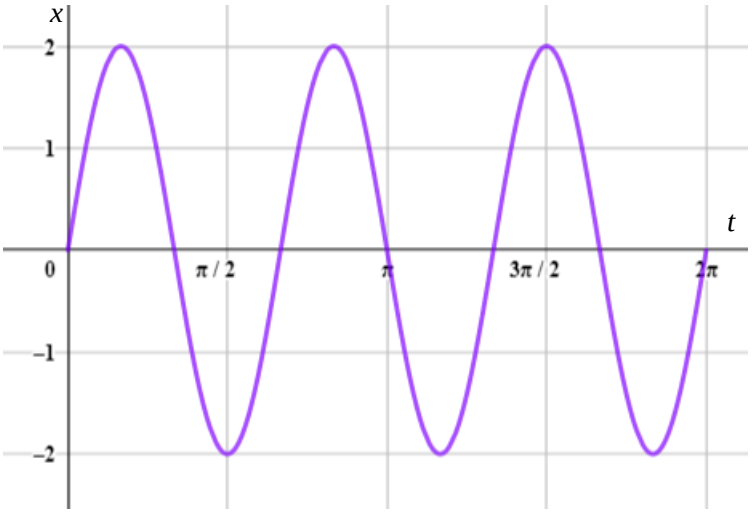
Solutions

Question 28. (3 marks)

A particle moves in a straight line. At time t seconds, its distance x metres from a fixed point O on the line is given by $x = 2\sin(3t)$

(i) Sketch $x = 2\sin(3t)$ for $0 \le t \le 2\pi$

Solution



Marks	Guideline
2	Correct response
1	Some correct working
Marker's Comment	Question was poorly done. Many responses could recognise the correct amplitude, but not the correct period. Many responses did not include markings for their intercepts/period and hence lost a mark. Some responses did not span the entire domain required or had incorrect/inconsistent markings for the period. The sine function is a smooth curve, NOT a straight line as some responses thought.

(ii) Find the first time when the particle is at rest.

Solution

The particle is first at rest when $t = \dfrac{\pi}{6}$

Marks	Guideline
1	Correct answer
Marker's Comment	Mostly well done but there were a lot of carried errors given here. It is at rest at its first stationary point, not at the axis. Pi/6 seconds is not the same as 30 seconds.

Question 29. (9 marks)

The queueing time, X minutes, of a teacher waiting on the phone with the Department of Education has a probability density function

$$f(x) = \begin{cases} \dfrac{3}{32}x(k-x) & 0 \le x \le k \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that the value of k is 4

$$\int_0^k \frac{3}{32}x(k-x)\,dx = 1$$

$$\frac{3}{32}\int_0^k kx - x^2\,dx = 1$$

$$\left[\frac{kx^2}{2} - \frac{x^3}{3}\right]_0^k = \frac{32}{3}$$

$$\left(\frac{k^3}{2} - \frac{k^3}{3}\right) - (0) = \frac{32}{3}$$

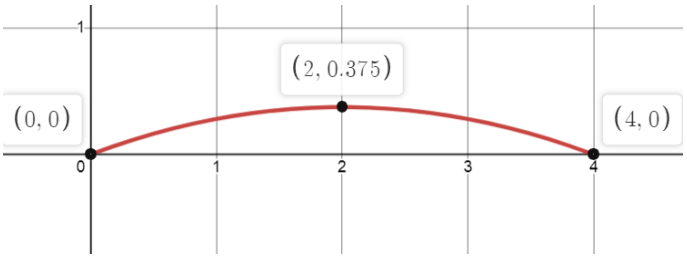
$$\frac{k^3}{6} = \frac{32}{3}$$

$$k = 4$$

Marks	Guideline
3	Correct response
2	One error (i.e. CTE)
1	Recognises integral = 1
Marker's Comment	Very poorly done. Many responses failed to recognise that the integral of a PDF over its bounds is equal to 1. Many responses ignored this fact or incorrectly stated that the PDF itself is equal to 1. There were many errors in working out to show that k=4 even when the integral was correctly applied.

Question 29 continued.

(ii) Sketch the probability density function $f(x)$



Marks	Guideline
2	Correct response
1	Some correct working
Marker's Comment	Poorly done. Many responses incorrectly assumed that this was a normal distribution and not a concave down parabola. A mark was given for the correct endpoints.

(iii) What is the mode of the probability density function?

Solution

From the graph the mode is $x = 2$

Marks	Guideline
1	Correct answer
Marker's Comment	Many carried errors given here.

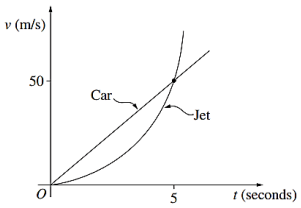
(iv) Find the probability that the phone will be answered in the first minute.

Solution

$$P(X \le 1) = \int_0^1 \frac{3}{32} x(4-x)\, dx$$
$$= \frac{3}{32}\left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_0^1$$
$$= \frac{3}{32}\left[\left(2(1)^2 - \frac{1^3}{3}\right) - (0)\right]$$
$$= \frac{3}{32}\left(\frac{5}{3}\right)$$
$$= \frac{5}{32}$$

Marks	Guideline
2	Correct response
1	Some correct working
Marker's Comment	Very poorly done. Once again, many responses failed to recognise that an integral from 0 to 1 was needed to find the probability. Some minor errors led to the wrong answer, but a mark was given for progress. Disappointingly,ome responses gave a negative probability or simply stated "unlikely" with no calculations.

Question 30 (8 marks)



A car and a jet plane race one another from rest down a runway. The car increases its speed v_1 at a constant rate, while the speed of the jet is given by $v_2 = 2t^2$. After 5 seconds the car and the jet have the same speed of 50 m/s, as shown on the graph.

(i) By considering the difference quotient, find the average acceleration of the jet during the 5th second.

Solution

$$\text{Average acceleration} = \frac{50 - 2(4)^2}{5 - 4}$$
$$= 18 \text{ m/s/s}$$

Marks	Guideline
2	Correct response
1	Some correct working
Marker's Comment	Almost all students found instantaneous rate of change instead of average rate of change. Read the question carefully, "**considering the difference quotient**" means you technically should have gotten 0 for doing instantaneous rate of change, however, due to overall poor performance

(ii) Find an equation for the speed v_1 of the car in terms of t.

Solution

$$v_1 = 10t$$

Marks	Guideline
1	Correct answer correctly.
Marker's Comment	Overall done well, some used x and y instead of V and t

(iii) For what times is the acceleration of the car greater than that of the jet?

Solution

$$a_{\text{car}} > a_{\text{jet}}$$
$$10 > 4t$$
$$t < \frac{5}{2} \text{ seconds}$$

Marks	Guideline
2	Correct response
1	Found derivative of velocity
Marker's Comment	Many students failed to recognise this inequality. Some used trial and error to find that the acceleration of the car to be less than the jet at 2 seconds and greater than at 3 seconds, hence said t = 3 instead of using an inequality to find it occurs at 2.5 seconds.

(iv) Who is winning the race after 5 seconds and by how many metres?

Solution

Marks	Guideline
3	Correct response (accept different forms of the function)
2	One error
1	One correct element

Distance of the car $= \int_0^5 10t \ dt$

$= \left[5t^2 \right]_0^5$

$= \left(5 \times 5^2 \right) - (0)$

$= 125$ metres

Therefore the car is winning by $\approx 125 - 83.3 \approx 41\frac{2}{3}$ m

Marker's Comment	Classic integration question which students should definitely recognise by now. Most students who recognised this got full marks, many students tried other approaches which were incorrect and hence got 0. Most students either got 0 or full marks. Some students thought "after 5 seconds" means in the 6th second.

Distance of jet $= \int_0^5 2t^2 \ dt$

$= \left[\frac{2t^3}{3} \right]_0^5$

$= \left(\frac{2 \times 5^3}{3} \right) - (0)$

≈ 83.3 metres

Question 31 (4 marks)

The lifetime of a particular make of LED bulb is normally distributed with mean of 5000 hours and standard deviation of 425 hours.

(i) Find the z score for a LED with a lifetime of 5374 hours.

Solution

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{5374-5000}{425}$$
$$= 0.88$$

Marks	Guideline
1	Correct answer
Marker's Comment	Straight forward question, mark awarded for no working, however, working was preferred. Overall done well.

(ii) By considering z scores and consulting the Normal Distribution table below, find the probability that a LED chosen at random has a lifetime between 4898 and 5374 hours. **3**

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891

Solution

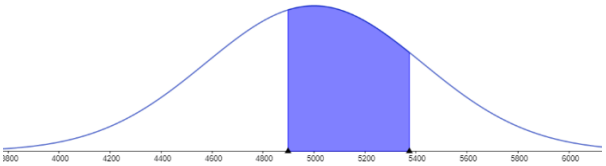
The z score for 4898 $z = \frac{4898-5000}{425}$
$= -0.24$

From the table

$$P(z \le -0.24) = 1 - P(z \le 0.24) \quad \text{(symmetrical)}$$
$$= 1 - 0.59483$$
$$= 0.40517$$

$$P(z \le 0.88) = 0.81057$$

$$\therefore \quad P(4898 \le x \le 5374) = P(z \le 0.88) - P(z \le -0.24)$$
$$= 0.81057 - 0.40517$$
$$= 0.4054$$



P(4898 ≤ X ≤ 5374) = 0.4054

Marks	Guideline
3	Correct response
2	One error (i.e. CTE)
1	One correct probability
Marker's Comment	Overall it was done ok by the cohort with partial marks given here and there, however, common errors occurred such as getting the z score for 4898 being 0.24 instead of –0.24 which gave carreid errors going forward. Almost all students setting out was difficult to follow. People would say things like 4898 = 0.24 = 0.59483 which is obviously nonsense. A handful of students read the value given for a z score of 0.42 instead.

Question 32. (5 marks)

The continuous random variable X has the following probability density function

$$f(x)=\begin{cases} a+bx & 0\le x\le 5 \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $10a+25b=2$

Solution

$$\int_0^5 a+bx\ dx=1$$
$$\left[ax+\frac{bx^2}{2}\right]_0^5=1$$
$$\left(5a+\frac{25b}{2}\right)-(0)=1$$
$$10a+25b=2$$

Marks	Guideline
2	Correct response
1	Recognized the integral of the function =1
Marker's Comment	Mostly well done. Better responses understood the area under a PDF would equal one and therefore integrated the function between 0 and 5.

(ii) If $E(X)=\frac{35}{12}$, find the values for a and b.

$$E(X)=\int x\times f(x)\ dx$$
$$E(X)=\int_0^5 x(a+bx)\ dx$$
$$\frac{35}{12}=\int_0^5 ax+bx^2\ dx$$
$$\frac{35}{12}=\left[\frac{ax^2}{2}+\frac{bx^3}{3}\right]_0^5$$
$$\left(\frac{25a}{2}+\frac{125b}{3}\right)-(0)=\frac{35}{12}$$
$$150a+500b=35$$

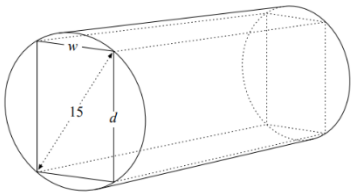
Marks	Guideline
3	Correct response (include by inspection if they confirm the result)
2	Substantial progress with an error
1	Recognized $E(X)=\int x\ f(x)\ dx$ or recognized the solution is found via simultaneous solutions
Marker's Comment	This was a difficult question. Some students correctly identified the idea of $x\times f(x)$ but incorrectly applied the *discrete* formula $E(X)=xp(x)$.

$$150a+500b=35\ \ ---(1)$$
$$10a+25b=2\ \ ---(2)$$

$$150a+375b=30\ \ ---(2A)$$
$$(1)-(2A)$$
$$125b=5$$
$$b=\frac{1}{25}$$

$$10a+25\left(\frac{1}{25}\right)=2$$
$$a=\frac{1}{10}$$

Question 33. (4 marks)



A rectangular beam of width w cm and depth d cm is cut from a cylindrical pine log as shown. The diameter of the cross-section of the log (and hence the diagonal of the cross-section of the beam) is 15 cm. The strength S of the beam is proportional to the product of its width and the square of its depth, so that $S = kwd^2$.

Find the maximum strength of the beam, showing why it is a maximum.

Solution

$$d^2 + w^2 = 15^2$$
$$d^2 = 225 - w^2$$

$$S = kwd^2$$
$$= kw\left(225 - w^2\right)$$
$$= 225kw - kw^3$$

$$\frac{dS}{dw} = 225k - 3kw^2$$

Max/min occurs when $\frac{dS}{dw} = 0$

$$225k - 3kw^2 = 0$$
$$w^2 = 75$$
$$w = \pm 5\sqrt{3}$$

Confirming a maximum at $w = 5\sqrt{3}$

$$\frac{d^2S}{dw^2} = -6kw$$

at $w = 5\sqrt{3}$

$$\frac{d^2S}{dw^2} = -6k\left(5\sqrt{3}\right)$$
$$< 0 \quad \text{since } k \text{ is a positive constant}$$
$$\Rightarrow \text{maximum turning point}$$

Note: w and d are positive (lengths) $\Rightarrow k > 0$ to give strength S as a positive

$$S_{\max} = k\left(5\sqrt{3}\right)(225 - 75)$$
$$= 750k\sqrt{3}$$

Marks	Guideline
4	Correct response
3	One error
2	Finding $w = 5\sqrt{5}$
1	Finding d^2 or w^2
Marker's Comment	Poorly answered. Many students should review the applications of calculus. Questions that ask for a maximum (or minimum) typically require you to differentiate and determine the nature of a stationary point. Try not to assume values, such as the angle the diagonal makes is 45°

Question 34 (6 marks)

The continuous random variable X has the probability density function $f(x)$ given by

$$f(x)=\begin{cases}\frac{1}{9}\left(x^2-2x+2\right) & 0\le x\le 3\\ \frac{1}{3} & 3<x\le 4\\ 0 & \text{otherwise}\end{cases}$$

(i) In the space provided, sketch the cumulative distribution function $F(x)$ 4

Solution

$$\begin{aligned}F(x)&=\int_0^x\frac{1}{9}\left(x^2-2x+2\right)dx\\&=\frac{1}{9}\left[\frac{x^3}{3}-x^2+2x\right]_0^x\\&=\frac{1}{9}\left[\left(\frac{x^3}{3}-x^2+2x\right)-0\right]\\&=\frac{1}{9}\left(\frac{x^3}{3}-x^2+2x\right)\end{aligned}$$

$$\begin{aligned}F(3)&=\frac{1}{27}\left(3^3-3\times3^2+6\times3\right)\\&=\frac{2}{3}\end{aligned}$$

The second section is a straight line joining $\left(3,\frac{2}{3}\right)$ to 1

Or alternatively $F(x)=\frac{2}{3}+\int_3^x\frac{1}{3}\,dx$

$$\begin{aligned}&=\frac{2}{3}+\left[\frac{x}{3}\right]_3^x\\&=\frac{2}{3}+\left[\frac{1}{3}x-1\right]\\&=\frac{1}{3}x-\frac{1}{3}\end{aligned}$$

Determining the features of $F(x)=\frac{1}{9}\left(\frac{x^3}{3}-x^2+2x\right)$ for $0\le x\le 3$

$$y=\frac{1}{9}\left(\frac{x^3}{3}-x^2+2x\right)$$

$y'=\frac{1}{9}\left(x^2-2x+2\right)$ $\Delta<0\Rightarrow$ no turning points

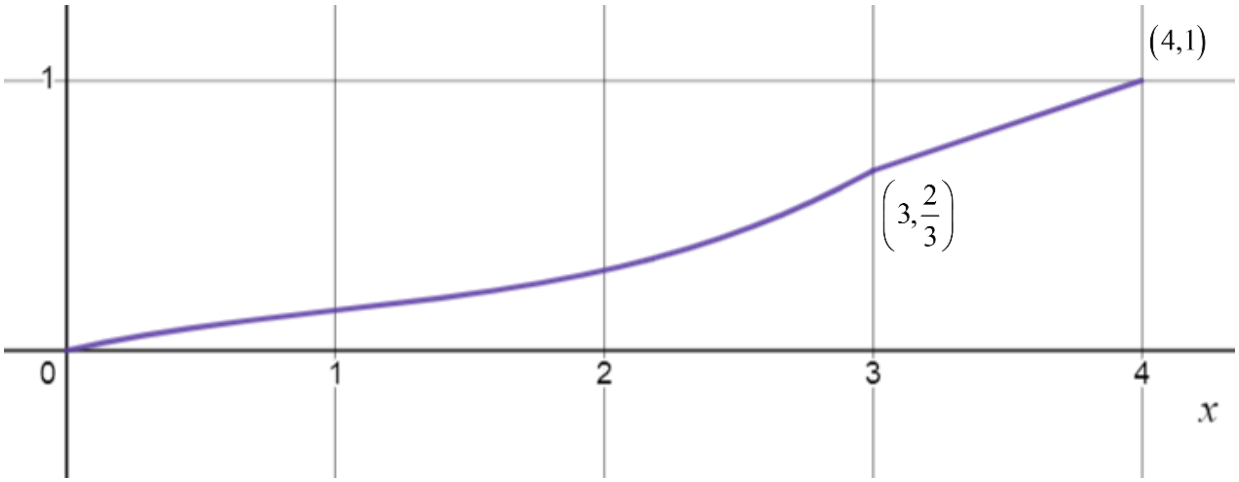
$y''=\frac{1}{9}(2x-2)$ $y''=0$ when $x=1$

Possible inflection at $x=1$

x	0	1	2
$\frac{d^2y}{dx^2}$	< 0	0	>0

Marks	Guideline
4	Correct response
3	One error
2	For finding the end of the first segment at $\left(3,\frac{2}{3}\right)$ with correct CDF
1	For finding $\frac{1}{9}\left(\frac{x^3}{3}-x^2+2x\right)$
Marker's Comment	Many students confused CDFs with PDF. No marks were awarded for drawing the PDF. Students who integrated part of the PDF attracted one mark. Better solutions recognised a CDF ends at 1 and is continuous. i.e. no break in the graph. Students had to clearly mark the points where the graph changed. i.e. $\left(3,\frac{2}{3}\right)$

Therefore there is a point of inflection at $x=1, y=\frac{4}{27}$



(ii) Hence, or otherwise, find the 75th percentile of X.

Solution

From the straight line segment found in part (i)

$$0.75=\frac{1}{3}x-\frac{1}{3}$$
$$2.25=x-1$$
$$x=3.25$$

Marks	Guideline
2	Correct response
1	Some progress
Marker's Comment	Some students recognised the CDF should be equal to 0.75 but used the wrong section of the CDF. Students should reflect upon their solution to identify errors. They could do this by drawing the percentile like shown below.

Alternatively from the graph